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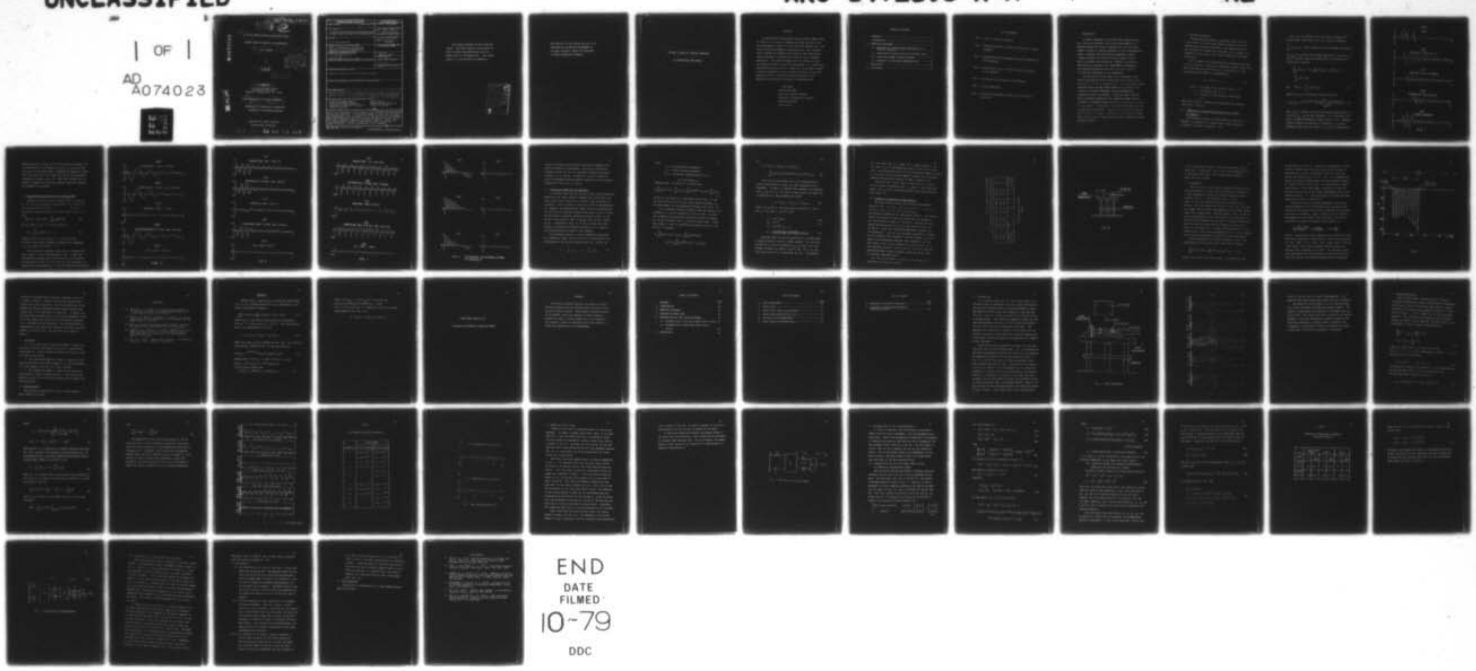

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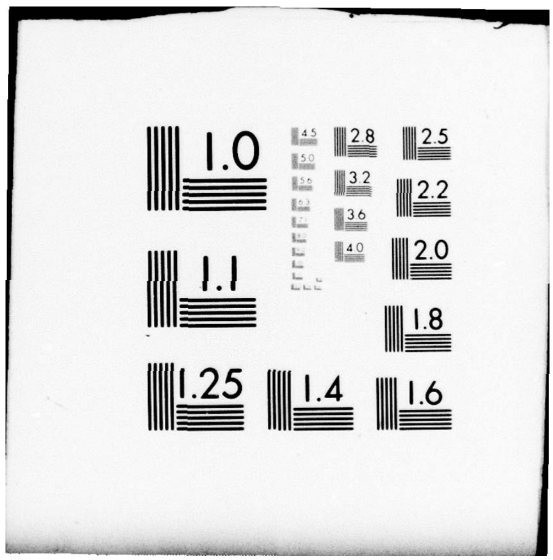
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A SYSTEM IDENTIFICATION METHODOLOGY USING DYNAMIC DATA FOR ANALYSIS OF MECHANISMS

9 Final Report. 1 Jan 77-30 Apr 79

by

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Natural Frequency
Curve Fitting
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Signals representing the force and displacement of the table of the M-139 machine gun during firing were analyzed. A constrained ARMA(2,1) model was fitted to the displacement data to explain the reason why the gun was jammed. Force data during and after firing were represented by the 4^{th} and 6^{th} order stochastic differential equation models. These models after decomposing yielded the dynamics of three subsystems; a supporting table, a recoil system and a bolt assembly.

This report consists of two technical papers. The first paper in the process of being accepted for publication and will appear soon in 'Technometrics'. The second paper is in the process of submission.

THE FINDINGS IN THIS REPORT ARE NOT TO BE CONSTRUED AS AN OFFICIAL DEPARTMENT OF THE ARMY POSITION, UNLESS SO DESIGNATED BY OTHER AUTHORIZED DOCUMENTS.

FITTING A SERIES OF IMPULSE RESPONSES

BY CONSTRAINED ARMA MODELS

ABSTRACT

A constrained autoregressive moving average (ARMA) model is used to fit a train of impulses response resulting from the displacement signal of firing the M-139 machine gun. The fitted constrained ARMA(2,1) model has only .38% residuals sum of squares as compared with 20.34% by the least square fitting assuming a deterministic model for a ten impulses experiment. The natural frequency of the system can also be estimated directly from the fitted model. In a particular experiment where the gun was jammed the fitted model yielded a natural frequency at 12.6 Hz while the actual firing frequency was 12.58. Hence the fitting technique can be employed to facilitate the design of mounting system.

Key Words

Constrained ARMA

Train of impulses response

Stochastic differential equation

Natural frequency

Curve fitting

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LIST OF FIGURES

1. INTRODUCTION

To model responses of a second order system excited by a series of impulses, such as the displacement of a mounting system for firing a machine gun, one can superimpose a series of impulse responses to fit a deterministic curve. Assuming the errors being independent normal random variables, the method of least squares can be used to estimate the parameters. However, in practice, the errors are generally not independent; consequently the estimated parameters will be inefficient leading to erroneous estimation of the parameters.

The objective of this paper is to introduce a time series technique which will facilitate the fitting of the response of a train of impulses data. The constrained autoregressive moving average (ARMA) model will be used to fit a series of impulse responses, and the mathematical justification will be derived. Since the model contains only the system parameters, the constrained ARMA fitting is easier to implement than the familiar deterministic curve fitting method. Furthermore, the constrained ARMA model can be physically interpreted. Real data of the M-139 machine gun firing will be used to demonstrate the constrained ARMA model fitting and its application to solve a mounting system failure problem.

2. MODELING TECHNIQUES

The deterministic model is applicable when the curve to be fitted is a known time function containing a set of unknown time-invariant parameters. The sampled values of this function at uniform time intervals, say Δ seconds, are assumed to be the sums of the "true" values and stationary white noise.

Let Y_t denotes the N observations at discrete time index t, and ε_t denotes a sequence of independent normal random variables with mean zero and variance $\sigma_a^2 (a_t \sim NID(0, \sigma_a^2)$, then the deterministic model is represented by:

$$Y_t = f(\underset{\sim}{\beta}, t) + \varepsilon_t \tag{1}$$

$f(\underset{\sim}{\beta}, t) \equiv$ the value of the curve at time $t_c = t\Delta$

$t_c \equiv$ continuous time; $t_c \geq 0$

$\underset{\sim}{\beta} \equiv$ p-vector of time-invariant parameters, $p \geq 0$

Note that if $\overline{f}(\underset{\sim}{\beta}, t_c)$ denotes the continuous time function, then $f(\underset{\sim}{\beta}, t) = \overline{f}(\underset{\sim}{\beta}, t\Delta)$.

2.1 Response of a Second Order System Due to a Train of Impulses

One frequently finds in practice that the curve to be modeled is a response of a second order system excited by an impulse or series of impulses. Let

ζ and ω_n be the damping ratio and natural frequency of a second order system whose input is a train of impulses $\sum_{k=1}^{M} c_k \delta(t_c - t_k)$, where M denotes the total number of impulses and c_k and t_k denote the strength and time of occurrence of the k^{th} impulse, then the response $\bar{f}(\underset{\sim}{\beta}, t_c)$ satisfies the differential equation

$$\frac{d^2}{dt_c{}^2}\bar{f}(\underset{\sim}{\beta},t_c) + 2\zeta\omega_n \frac{d}{dt_c}\bar{f}(\underset{\sim}{\beta},t_c) + \omega_n{}^2\bar{f}(\underset{\sim}{\beta},t_c) = \sum_{k=1}^{M} c_k\delta(t_c-t_k),$$

and

$$\bar{f}(\underset{\sim}{\beta},t_c) = \sum_{k=1}^{M} c_k G(t_c-t_k) \tag{2}$$

where $G(t_c-t_k)$ is the Green's function given by

$$G(t_c-t_k) = e^{-\zeta\omega_n(t_c-t_k)} \frac{\sin \omega_n\sqrt{1-\zeta^2}\,(t_c-t_k)u(t_c-t_k)}{\omega_n\sqrt{1-\zeta^2}}. \tag{3}$$

Here $u(t_c-t_k)$ denotes the unit step function whose value is unity for $t_c \geq t_k$ and zero otherwise. An illustration of a train of impulses response is shown in Fig. 1 for 3 impulses. Figure 1a shows the impulses of strengths c_1, c_2 and c_3 disturbing the system at times t_1, t_2 and t_3 respectively.

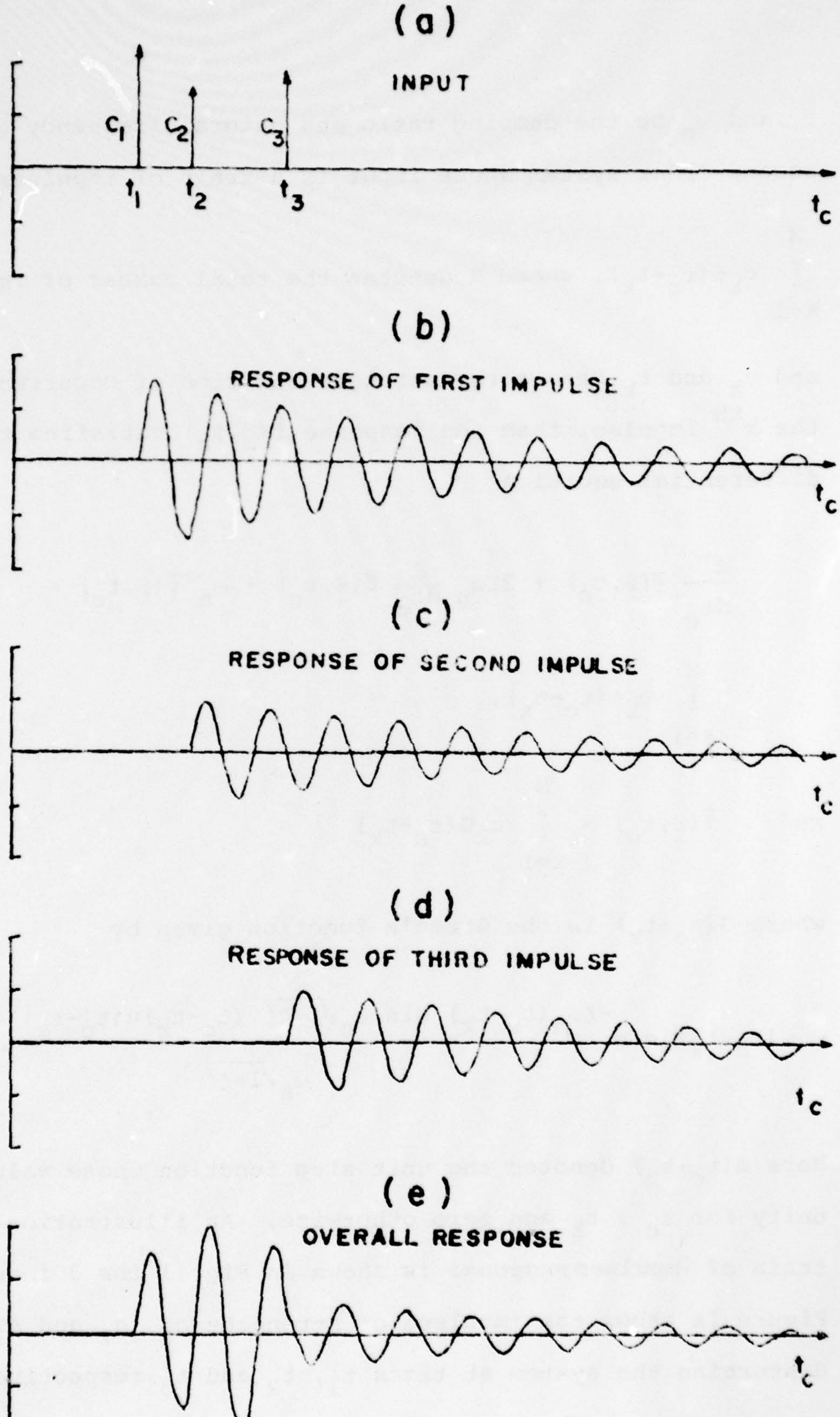
(a)
INPUT
c_1
c_2
c_3
t_1
t_2
t_3
t_c
(b)
RESPONSE OF FIRST IMPULSE
t_c
(c)
RESPONSE OF SECOND IMPULSE
t_c
(d)
RESPONSE OF THIRD IMPULSE
t_c
(e)
OVERALL RESPONSE
t_c

FIG. 1

Computed by Eq. 9, Figs. 1b, 1c and 1d show the responses of the system due to the impulses $c_1\delta(t_c-t_1)$, $c_2\delta(t_c-t_2)$ and $c_3\delta(t_c-t_3)$ in the same order. Assuming the system is linear, then the output $f(\underset{\sim}{\beta},t)$ is simply the sum of the individual responses, as shown in Fig. 1e. Note that neither the impulses' strengths nor the time periods between the impulses are assumed to be uniform.

2.2 Deterministic Curve Fitting of Discrete Data

Modeling of the continuous response $\bar{f}(\underset{\sim}{\beta},t_c)$ given by Eq. (2) can be conveniently implemented if it is observed at a uniform time interval. If $f(\underset{\sim}{\beta},t)$ denotes the observed values at discrete time t=1,2,...,N, then

$$f(\underset{\sim}{\beta},t) = f(\underset{\sim}{\beta},t_c/\Delta) = \sum_{k=1}^{M} c_k G(t\Delta-t_k) \tag{4}$$

and the model of Eq. (1) can be written as

$$Y_t = \sum_{k=1}^{M} c_k G(t\Delta-t_k) + \varepsilon_t \tag{5}$$

Assuming that ε_t's are NID$(0,\sigma_\varepsilon^2)$, one can use the nonlinear least squares method to estimate the parameters ζ, ω_n, c_k and t_k for k=1,2,..., M.

As illustrative examples, three sets of response data were fitted. It will be discussed in Sec. 3 that the digitized experimental data from firing the M-139 machine gun displayed in Figs. 2a, 3a and 4a are the responses of second-order systems due to 1, 3 and 10 impulses respectively.

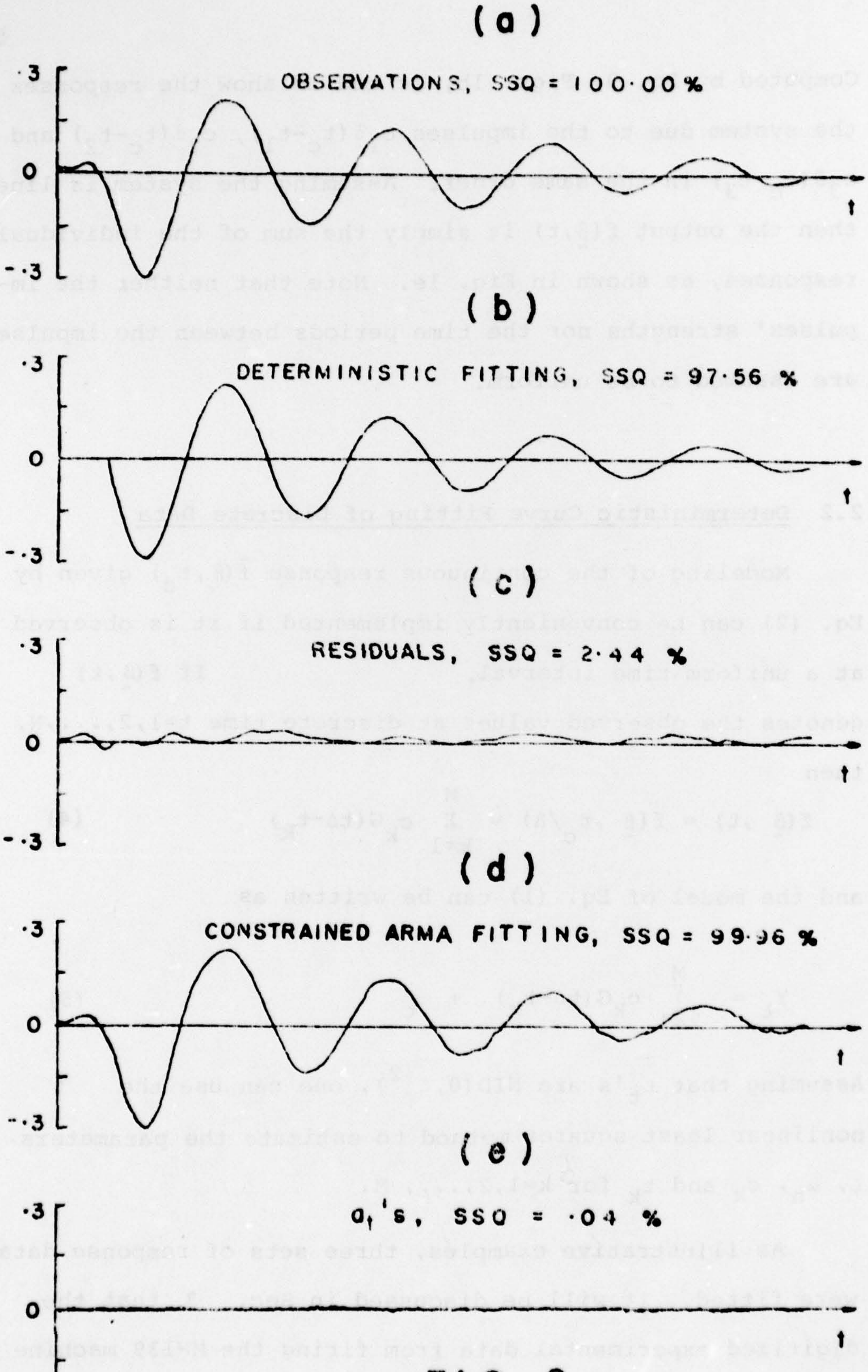
(a)
·3
OBSERVATIONS, SSQ = 100·00 %
0
t
-·3
(b)
·3
DETERMINISTIC FITTING, SSQ = 97·56 %
0
t
-·3
(c)
·3
RESIDUALS, SSQ = 2·44 %
0
t
-·3
(d)
·3
CONSTRAINED ARMA FITTING, SSQ = 99·96 %
0
t
-·3
(e)
·3
a_t's, SSQ = ·04 %
0
t
-·3

FIG. 2

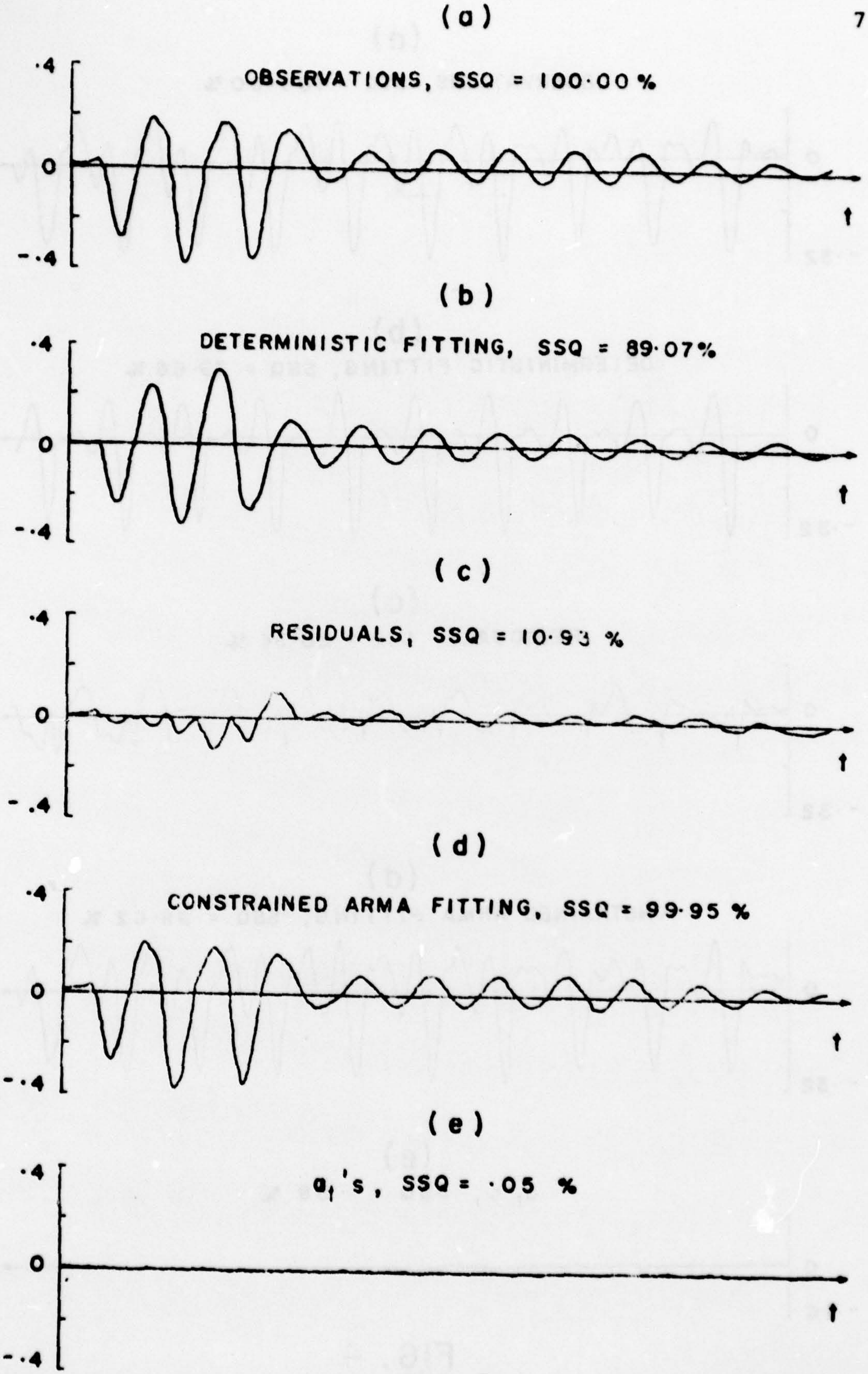
(a)
OBSERVATIONS, SSQ = 100·00 %
·4
0
-·4
t
(b)
DETERMINISTIC FITTING, SSQ = 89·07%
·4
0
-·4
t
(c)
RESIDUALS, SSQ = 10·93 %
·4
0
-·4
t
(d)
CONSTRAINED ARMA FITTING, SSQ = 99·95 %
·4
0
-·4
t
(e)
a_t's, SSQ = ·05 %
·4
0
-·4
t

FIG. 3

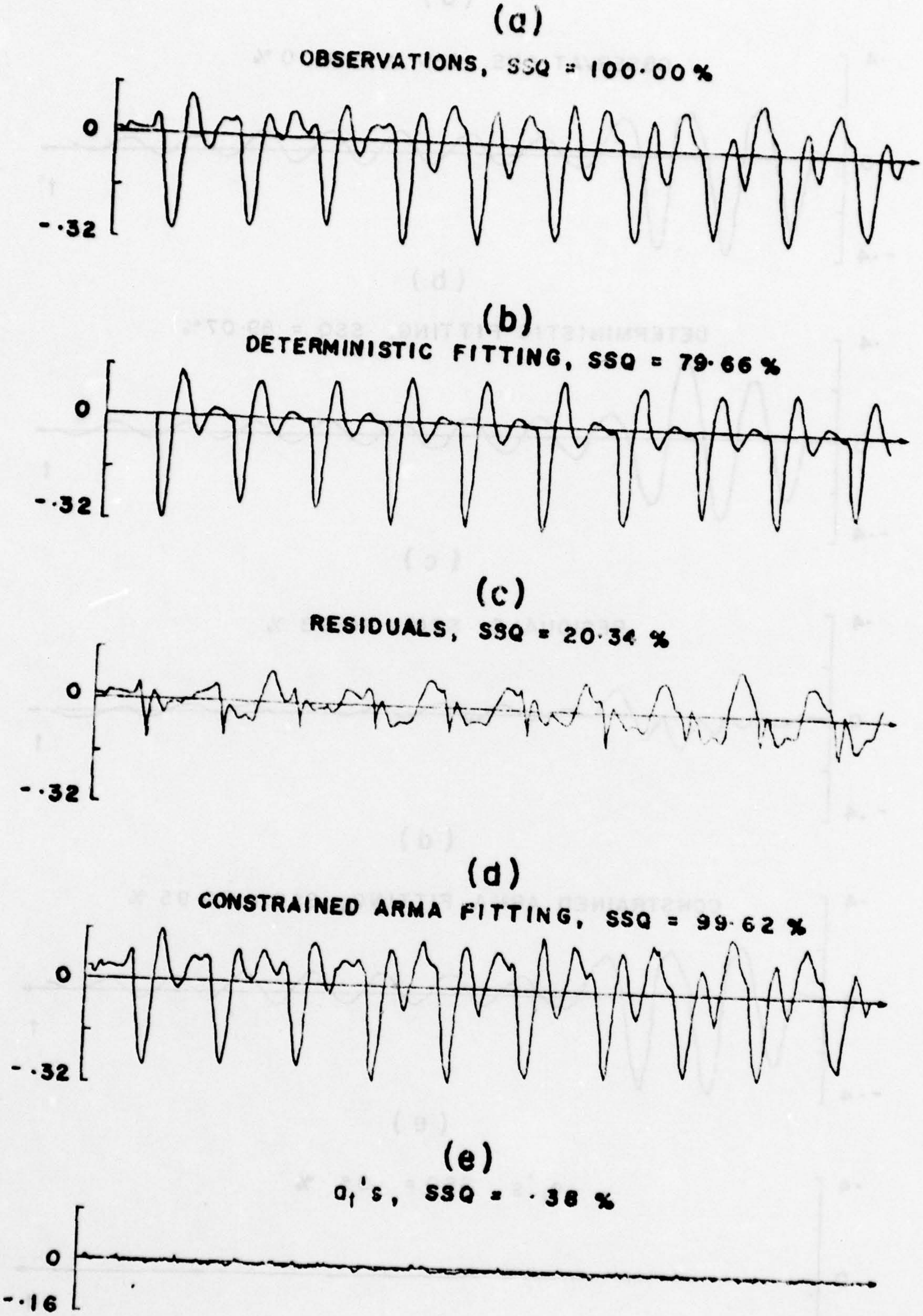
(a)
OBSERVATIONS, SSQ = 100·00 %
0
-·32
(b)
DETERMINISTIC FITTING, SSQ = 79·66 %
0
-·32
(c)
RESIDUALS, SSQ = 20·34 %
0
-·32
(d)
CONSTRAINED ARMA FITTING, SSQ = 99·62 %
0
-·32
(e)
a_t's, SSQ = ·38 %
0
-·16

FIG. 4

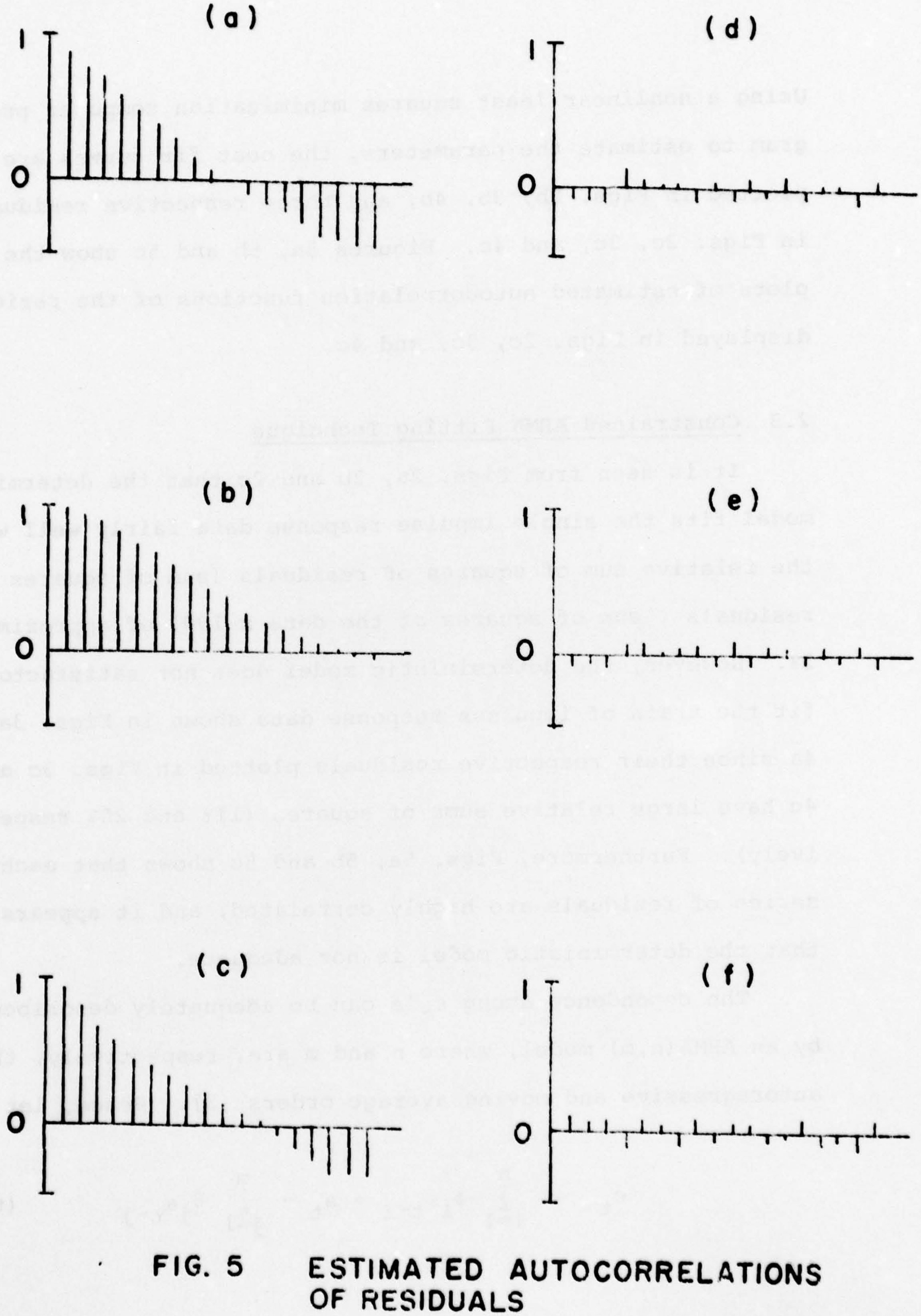

FIG. 5 ESTIMATED AUTOCORRELATIONS OF RESIDUALS

Using a nonlinear least squares minimization computer program to estimate the parameters, the best fit curves are plotted in Figs. 2b, 3b, 4b, and three respective residuals in Figs. 2c, 3c, and 4c. Figures 5a, 5b and 5c show the plots of estimated autocorrelation functions of the residuals displayed in Figs. 2c, 3c, and 4c.

2.3 Constrained ARMA Fitting Technique

It is seen from Figs. 2a, 2b and 2c that the deterministic model fits the single impulse response data fairly well with the relative sum of squares of residuals (sum of squares of residuals ÷ sum of squares of the data x 100) of approximately 2%. However, the deterministic model does not satisfactorily fit the train of impulses response data shown in Figs. 3a and 4a since their respective residuals plotted in Figs. 3c and 4c have large relative sums of squares (11% and 20% respectively). Furthermore, Figs. 5a, 5b and 5c shows that each series of residuals are highly correlated, and it appears that the deterministic model is not adequate.

The dependence among ε_t's can be adequately described by an ARMA(n,m) model, where n and m are, respectively, the autoregressive and moving average orders [3]. Hence, let

$$\varepsilon_t = \sum_{i=1}^{n} \phi_i \varepsilon_{t-i} + a_t - \sum_{j=1}^{m} \theta_j a_{t-j} \qquad (6)$$

where

ϕ_i's = autoregressive parameters

θ_j's = moving average parameters

a_t's = orthogonal decompositions of ε_t's,

a_t's are NID$(0,\sigma_a{}^2)$

Combining Eqs. (5) and (6), it follows that

$$Y_t - \sum_{k=1}^{M} c_k G(t\Delta - t_k) = \sum_{i=1}^{n} \phi_i [Y_{t-i} - \sum_{k=1}^{M} c_k G(\overline{t-i}\Delta - t_k)] + a_t - \sum_{j=1}^{m} \theta_j a_{t-j} \tag{7}$$

Since a_t's are NID$(0,\sigma_a{}^2)$, the model given by Eq. (7) can be fitted using the non linear least square estimators in the same manner as the deterministic model given by Eq. (1).

It is frequently found in practice that the order of the residuals ε_t's can be assumed to be two or less. In fact, the estimated autocorrelation functions plotted in Figures 5a, 5b and 5c suggest that ε_t's are of second-order. Therefore, model (7) may be simplified by assuming that n=2, m=1 and Eq. (7) becomes:

$$Y_t - \sum_{k=1}^{M} c_k G(t\Delta - t_k) = \phi_1 [Y_{t-1} - \sum_{k=1}^{M} c_k G(\overline{t-1}\Delta - t_k)]$$

$$+ \phi_2 [Y_{t-2} - \sum_{k=1}^{M} c_k G(\overline{t-2}\Delta - t_k)] + a_t - \theta_1 a_{t-1}$$

or

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + a_t - \theta_1 a_{t-1}$$

$$+ \sum_{k=1}^{M} c_k [G(t\Delta - t_k) - \phi_1 G(\overline{t-1}\Delta - t_k) - \phi_2 G(\overline{t-2}\Delta - t_k)] \qquad (8)$$

Note that fitting model (8) to the response data is rather inconvenient, as there are 2M+3 parameters to be estimated. In order to simplify this problem it is assumed that ε_t and $G(t\Delta - t_k)$ are from the same system, and model (8) is simplified to (see appendix for proof)

$$Y_t = \phi_1 Y_{k-1} + \phi_2 Y_{t-2} + a_t - \theta_1 a_{t-1} \qquad (9)$$

where ϕ_1, ϕ_2 and θ_1 are functionally related to ζ and ω_n. Let $a = \zeta\omega_n$, and $b = \omega_n\sqrt{1-\zeta^2}$, then

$$\phi_1 = 2e^{-a\Delta} \cos b\Delta \qquad (10)$$

$$\phi_2 = -e^{-2a\Delta} \qquad (11)$$

$$\theta_1 = -P \pm \sqrt{P^2 - 1}, |\theta_1| < 1,$$

$$2P = \frac{b \sinh 2a\Delta - a \sin 2b\Delta}{a \sin b\Delta \cosh a\Delta - b \sinh a\Delta \cos b\Delta} \qquad (12)$$

Although model (9) has an appearance of the ARMA model, it differs from the latter in many respects. The most pronounced one is that the parameters ϕ_1, ϕ_2 and θ_1 are functionally related, as clearly shown by Eqs. (10), (11) and (12), while those of the ARMA model are not. Furthermore, Eq. (11) shows that ϕ_2 in model (9) is always negative, and the value of ϕ_2 in the ARMA model does not have this restriction. To distinguish model (9) from the general ARMA(2,1) model, it is called the constrained ARMA(2,1) model [4] or uniformly sampled autoregressive process of order 2 (USA(2)) [3].

Note that Eqs. (10,11, and 12) show that ϕ_1, ϕ_2 and θ_1 can be uniquely determined from ζ, ω_n and Δ. Hence the parameter ζ and ω_n can be estimated directly from digitized data using model (9).

2.4 Examples of Constrained ARMA Modeling

The data plotted in Figs. 2a, 3a and 4a was modeled using Eq. (9). The least squares estimators of ζ and ω_n are obtained by Eqs. (9-12), and the continuous

and discrete parameters are tabulated in Table 1. The fitted curves are plotted in Figs. 2d, 3d and 4d; the respective a_t's are plotted in Figs. 2e, 3e and 4e. Finally, the estimated autocorrelations of the a_t's are shown in Figs. 5d, 5e and 5f. Since the constrained ARMA model contains only two parameters, ζ and ω_n, the estimation procedure is simple and the computer time required is greatly reduced. For example, the approximated computer time required to fit the data shown in Fig. 4a was reduced by a factor of 10. It is also apparent from Figs. 2e, 3e, 4e, 5d, 5e and 5f that the constrained ARMA fitting technique yields the curves that fit the data remarkably well.

The constrained ARMA(2,1) model is used because its

parameters are directly related

Table 1

Results of ARMA Modeling

Data in Figures	ϕ_1	ϕ_2	θ_1	ζ	f_n*	Relative SSQ**
2a	1.961	-.974	-.268	.116	12.0	0.04%
3a	1.958	-.972	-.269	.119	12.6	0.05%
4a	1.862	-.918	-.269	.176	25.7	0.38%

*f_n denotes natural frequency in Hertz, $f_n = \omega_n/2\pi$.

**Relative sum of squares of a_t's.

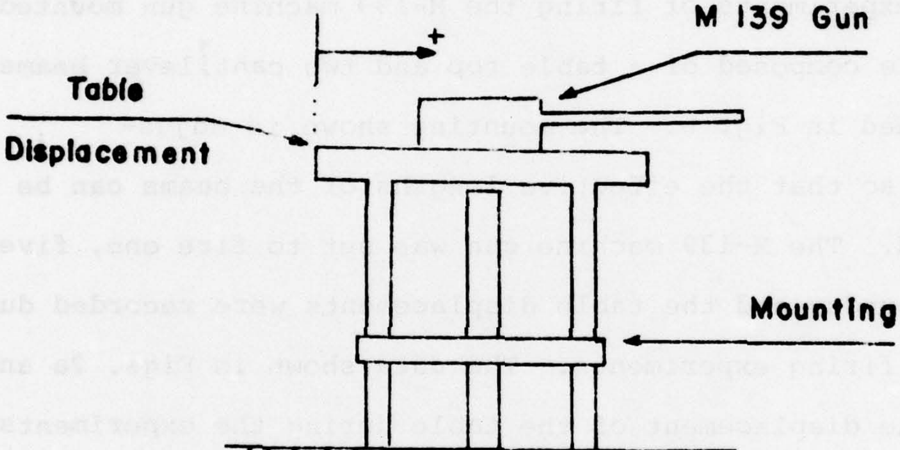

FIG. 6

to the system under studies, and these parameters can be physically interpretted. But if the system parameters are of secondary importance, the (unconstrained) ARMA(2,1) model can also be used.

3. APPLICATION

The three sets of data used as examples previously are from experiments of firing the M-139 machine gun mounted on a table composed of a table top and two cantilever beams sketched in Fig. 6. The mounting shown is adjustable so that the effective lengths of the beams can be varied. The M-139 machine gun was set to fire one, five and ten rounds, and the table displacements were recorded during these firing experiments. The data shown in Figs. 2a and 4a are the displacement of the table during the experiments of firing the M-139 machine gun one and ten rounds respectively. There are occasions when the M-139 machine gun was set to fire 5 or 10 rounds, but it failed consistently after 3 successive fires; and the data displayed in Fig. 3a is an example of the table displacements obtained from such experiments.

The displacements of the table sketched in Fig. 6 are characterized by the second order differential equation:

$$\frac{d^2}{dt_c^{\,2}} \bar{f}(\underset{\sim}{\beta},t_c)+2\zeta\omega_n \frac{d}{dt_c} \bar{f}(\underset{\sim}{\beta},t_c)+\omega_n^{\,2}\bar{f}(\underset{\sim}{\beta},t_c) = I(t_c)$$

where $I(t_c)$ is the input force, and ζ, ω_n and $\bar{f}(\underset{\sim}{\beta},t_c)$ are as described in Section 2. During the experiments of firing the M-139 machine gun, the input force can be closely approximated by a series of impulses, each with occurring time t_k and strength c_k, $k = 1,2,\ldots,M$. It is assumed that during the course of experiments the table vibrates randomly, hence the white noise, $Z(t_c)$, is specified as an additional input. It has been shown in Section 2.3 that the discrete observations of the table displacement Y_t's can be modeled using Eq. (9), and the results of constrained ARMA modeling were given in Section 2.4.

During the course of firing experiments, the average firing frequency of the M-139 machine gun, defined as the inverse of the average time intervals between two successive fires can be estimated. Figure 7 shows a plot of relative displacement of the receiver and the cradle of the M-139 gun used to estimate the average firing period, i.e.

$$\frac{(2 + \frac{5}{16})\ \text{inches}}{(10-1)\ \text{fires}} \times \frac{2\ \text{secs}}{6.5\ \text{inches}} = .0791\ \frac{\text{sec}}{\text{fire}} \quad .$$

similar calculations (using computer) were carried out using data from 10 additional experiments to obtain firing periods of .0797, .0809, .0797, .0782, .0792, .0809, .0801, .0794 and .0783 second. From these estimated firing periods, the average firing frequency is 12.58 Hz. For the table mountings associated with the experiments in which the M-139 machine gun failed, the constrained ARMA modeling technique

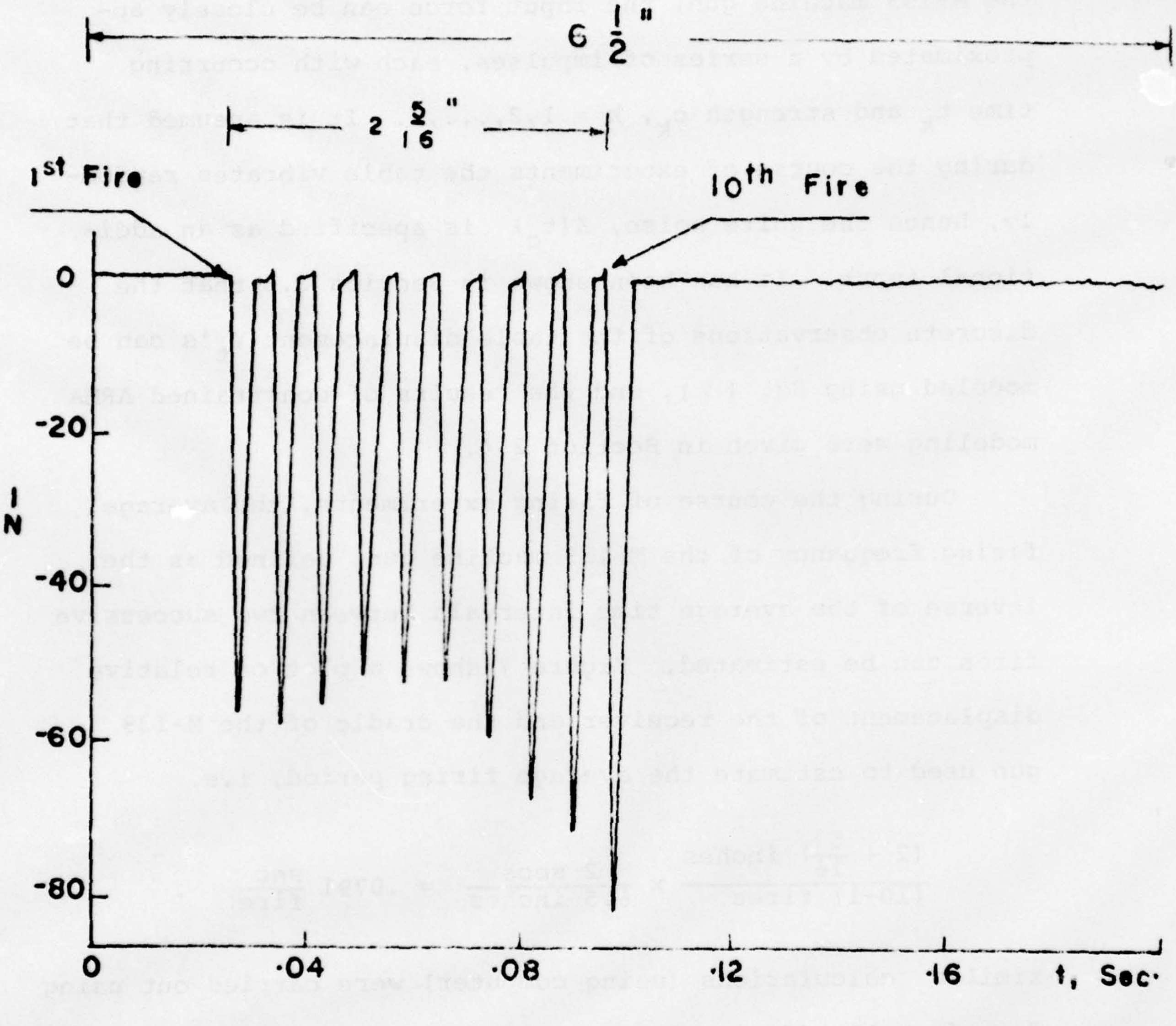

FIG. 7

yielded an estimated table's natural frequency of 12.6 Hz as shown in Table 1. Noting this, one can conclude that during the firing experiments, the M-139 machine gun failed because its firing frequency and the table's natural frequency were of the same order of magnitude. It should also be noted that the table used in the firing experiments which the M-139 machine gun did not fail had different mountings (i.e., different spring constants). The estimated natural frequencies of the table were different from the firing frequency of 12.58 Hz; e.g., 12.0 and 25.7 Hz as shown in Table 1.

5. CONCLUSION

1. It is shown that a constrained ARMA(2,1) model can be used to fit the experimental responses of a second-order system due to a train of impulses assuming the noise is also an output of the system.

2. The constrained ARMA(2,1) model is superior to the signal plus white noise model because of (i) less parameters, (ii) less computer time and (iii) less residuals.

3. The constrained ARMA(2,1) model can be physically interpreted in terms of damping factor, ζ, and natural frequency, ω_n; hence it can be used in designing the machine gun mounting system.

6. ACKNOWLEDGEMENT

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APPENDIX

Suppose that ε_t and $G(t\Delta-t_k)$ are from the same system; i.e., ε_t is a uniform sampling of $\varepsilon(t_c)$ governed by a stochastic differential equation

$$\frac{d^2}{dt_c^{\,2}}\,\varepsilon(t_c)+2\zeta\omega_n \frac{d}{dt_c}\,\varepsilon(t_c)+\omega_n^{\,2}\,\varepsilon(t_c) = Z(t_c) \qquad \text{A.1}$$

where $Z(t_c)$ is the white noise process with properties $E(\varepsilon(t_c)) = 0$, $E(Z(t_c)Z(t_c+\tau)) = \sigma_z^{\,2}\delta(\tau)$. The discrete process ε_t is represented by [1,2,4],

$$\varepsilon_t = \phi_1\varepsilon_{t-1} + \phi_2\varepsilon_{t-2} + a_t - \theta_1 a_{t-1} \qquad \text{A.2}$$

where ϕ_1, ϕ_2 and θ_1 are as defined by Eqs. (10), (11) and (12). Also $G(t\Delta-t_k)$ defined by Eq. (3) can be written as

$$G(t\Delta-t_k) = e^{-a(t\Delta-t_k)}\sin(b(t\Delta-t_k))U(t\Delta-t_k)/b \qquad \text{A.3}$$

Suppose that $(t-2)\Delta-t_k \geqslant 0$, then $U(t\Delta-t_k) = 1$, and using $\phi_1 = 2e^{-a\Delta}\cos b\Delta$, $\phi_2 = -e^{-2a\Delta}$ and Eq. A.3, it can be shown by algebra that

$$G(t\Delta-t_k) - \phi_1\, G(\overline{t-1}\Delta-t_k) - \phi_2\, G(\overline{t-2}\Delta-t_k) = 0 \qquad \text{A.4}$$

Since $G(t\Delta - t_k) = 0$ for $t\Delta - t_k < 0$, it follows that $G(t\Delta - t_k) - \phi_1 G(\overline{t-1}\Delta - t_k) - \phi_2 G(\overline{t-2}\Delta - t_k) = 0$ when $t\Delta - t_k < 0$ or $(t-2)\Delta - t_k \geq 0$. Hence Eq. (8) can be closely approximated by Eq. (9); i.e.,

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + a_t - \theta_1 a_{t-1} \quad .$$

M-139 FORCE ANALYSIS BY

STOCHASTIC DIFFERENTIAL EQUATION MODELS

ABSTRACT

Force data recorded during an experiment of firing the M-139 machine gun are analyzed using stochastic differential equation models. These models represent overall M-139 systems, which are composed of three systems: a supporting table, a recoil system and a bolt assembly. The analysis is made to decompose these force models to obtain the parameters of the subsystems.

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1. Introduction

In a previous paper [4], the table displacement data resulting from the experiments of firing the M-139 machine gun has been analyzed using a constrained ARMA(2,1) model. The model was fitted under the assumptions that the signals are a train of impulse responses and the noise is due to random shocks. The constrained ARMA(2,1) model fits the table displacement data well, as the residuals variances are less than 0.4% of the variances of the data. The damping ratio and natural frequency of the table were estimated from which it was shown that the M-139 machine gun jammed when its firing frequency coincided with the table's natural frequency.

Physically a firing experiment contains not only the table but also the M-139 machine gun; yet, the analysis of the table displacement data did not reveal the gun dynamics. By examining a sketch of the firing experiment shown in Fig. 1, it can be seen that the table displacement data was recorded afar from the gun. Consequently, one expects little or no effect of the gun dynamics on the table displacements. During these firing experiments, the reaction forces at the mounting of the gun to the table were also recorded. For an experiment of firing ten sucessive rounds of the M-139 machine gun, the recorded forces, taken at the four corners of the rectangular-shape gun's base, are shown in Figs. (2A-2D). Since these forces were recorded much

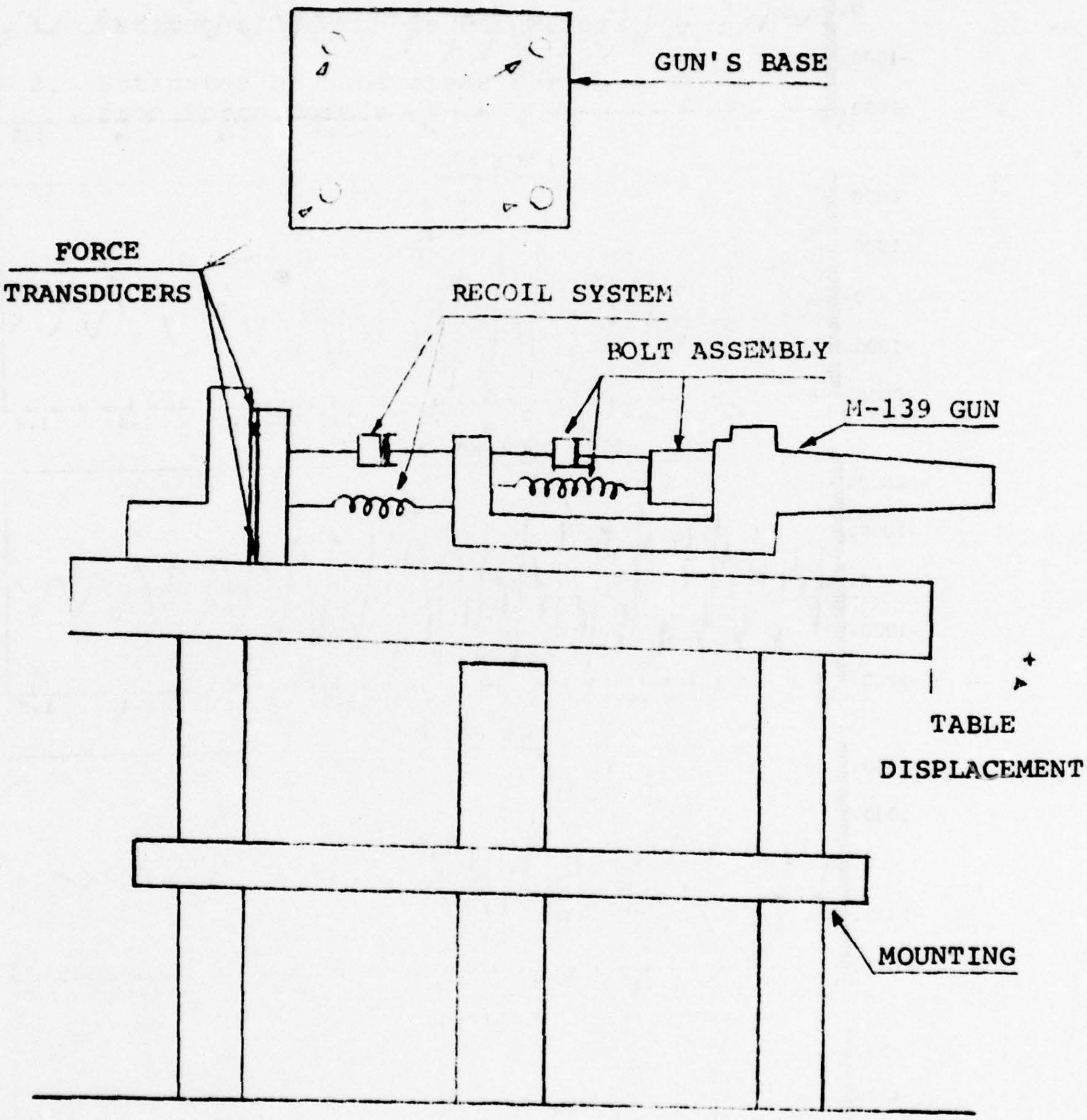

FIG. 1 FIRING EXPERIMENT

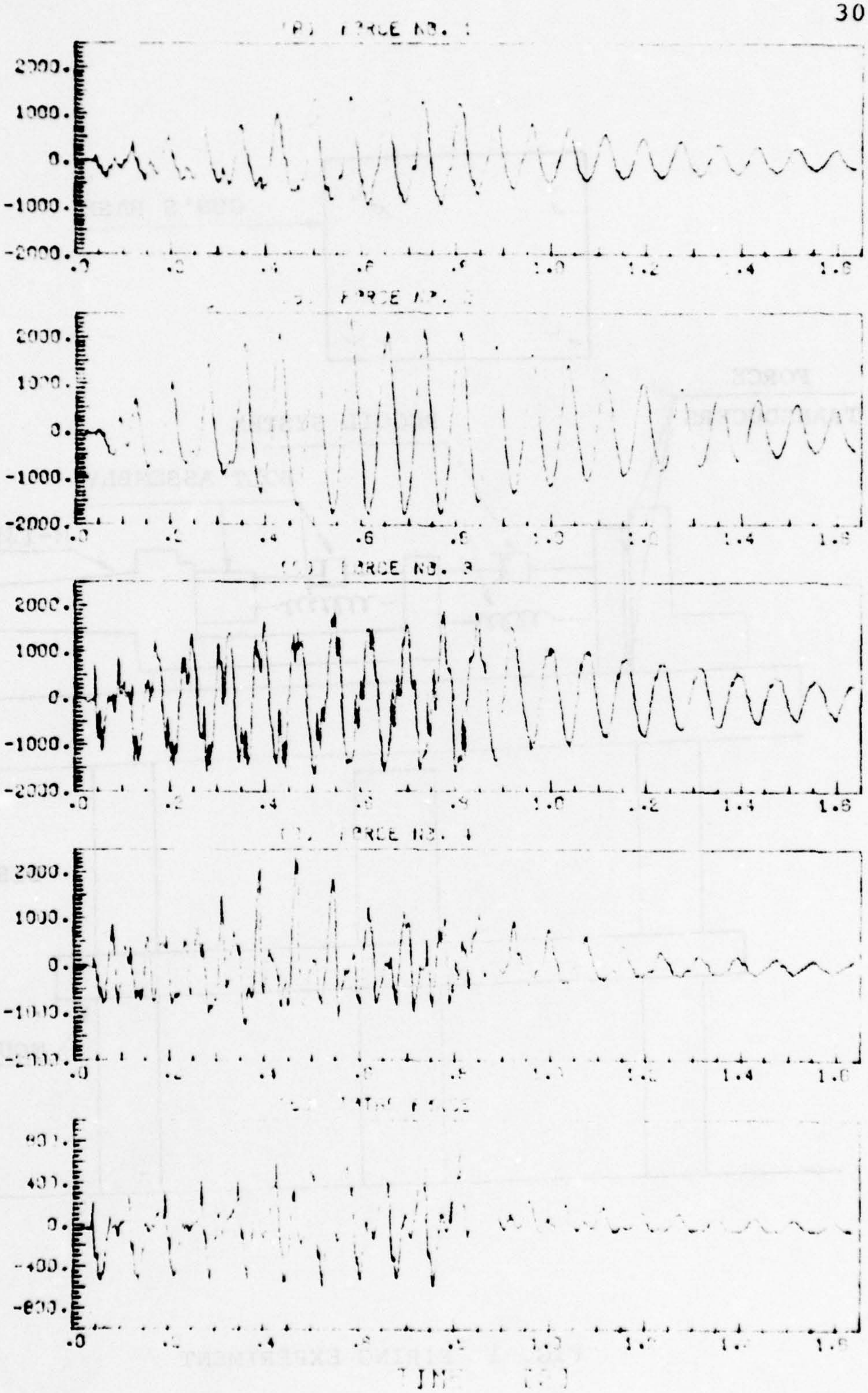

FIG. 2 FORCES DATA

closer to the gun than the table displacements, it is expected that these forces data contain the gun's dynamics.

The objective of this paper is to fit the force data into linear stochastic differential equation models. The orders of these models are expected to be higher than two, as the force data contain the gun's as well as the table's dynamics. Since the differential equation fitted to the force data describes a coupled system, a method of decomposition is developed in order to interpret the model in the form of the firing experiments' subsystems.

2. Modeling Technique

The digitized force data was fitted to obtain a stochastic differential equation model which is formulated by letting t_c denote a continuous time, and $Z(t_c)$ denote the white noise process with properties $E(Z(t_c)) = 0$ and $E(Z(t_c)Z(t_c+s)) = \sigma_z^2\,\delta(s)$, where $\delta(s)$ is the Dirac delta function. The continuous autoregressive moving average process of orders n and m (AM(n,m)), denotes by $X(t_c)$, can be defined in terms of symbolic derivatives as

$$\frac{d^n X(t_c)}{dt_c^n} + \sum_{i=1}^{n} \alpha_{n-i} \frac{d^{n-i}}{dt_c^{n-i}} X(t_c)$$

$$= Z(t_c) + \sum_{j=1}^{m} b_j \frac{d^j}{dt_c^j} Z(t_c) \tag{1}$$

where the coefficients α_i's and b_j's are real.

Let μ_i, $i=1, 2, \ldots, n$ be the characteristic roots; i.e., μ_i's are solutions of the equation

$$\mu^n + \alpha_{n-1}\mu^{n-1} + \ldots + \alpha_1\mu + \alpha_0 = 0 \tag{2}$$

The required conditions for process (1) to be stationary are $m<n$ and the real parts of all μ_i's be negative [3].

The autocovariance function of $X(t_c)$ is

$$\gamma(s) = E(X(t_c)X(t_c+s)) = \sum_{j=1}^{n} c_j \exp(\mu_j|s|) \tag{3}$$

where

$$c_j = \sigma_z^2 \, N(\mu_j) N(-\mu_j) \bigg/ \left[2\mu_j \prod_{k=1,\ k\neq j}^{n} (\mu_j^2 - \mu_k^2)(-1)^n \right]$$

$$N(\mu_j) = 1 + b_1\mu_j + b_2\mu_j^2 + \ldots + b_m\mu_j^m \tag{4}$$

When process (1) is sampled at a uniform discrete time interval, say Δ seconds, the resultant discrete process, X_t, can be represented by a uniformly sampled autoregressive moving average model (USAM) of the form [2,3,4]

$$X_t - \sum_{i=1}^{n} \phi_i X_{t-i} = a_t - \sum_{j=1}^{n-1} \theta_j a_{t-j} \tag{5}$$

where a_t is a discrete white noise process with variance σ_a^2, and ϕ_i's and θ_j's which are functionally related to α_i's and b_j's defined by [2,3]

$$\prod_{i=1}^{n} (1-e^{\mu_i \Delta} B) = \prod_{i=1}^{n} (1 - \lambda_i B) = 1 - \sum_{i=1}^{n} \phi_i B^i \tag{6}$$

and if ν_j's are the n-1 invertible roots ($|\nu_j|<1$) of the polynomial

$$P(B) = \sum_{j=1}^{n} c_j (1-\lambda_j^2) \prod_{k=1,k\neq j}^{n} (1-\lambda_k B)(1-\lambda_k B^{-1}) \tag{7}$$

then

$$\prod_{j=1}^{n-1} (1-\nu_j B) = 1 - \sum_{j=1}^{n-1} \theta_j B^j \tag{8}$$

The parameters of the continuous process (1) can be estimated from a set of uniformly sampled data using maximum likelihood estimators or approximated by least squares estimators [2]. Therefore, for a given set of uniformly sampled data, the sum of squares of a_t's in model (5) is expressed in terms of continuous parameters. Using this procedure, the sum of squares of a_t's can be minimized to obtain the direct estimates of the continuous parameters.

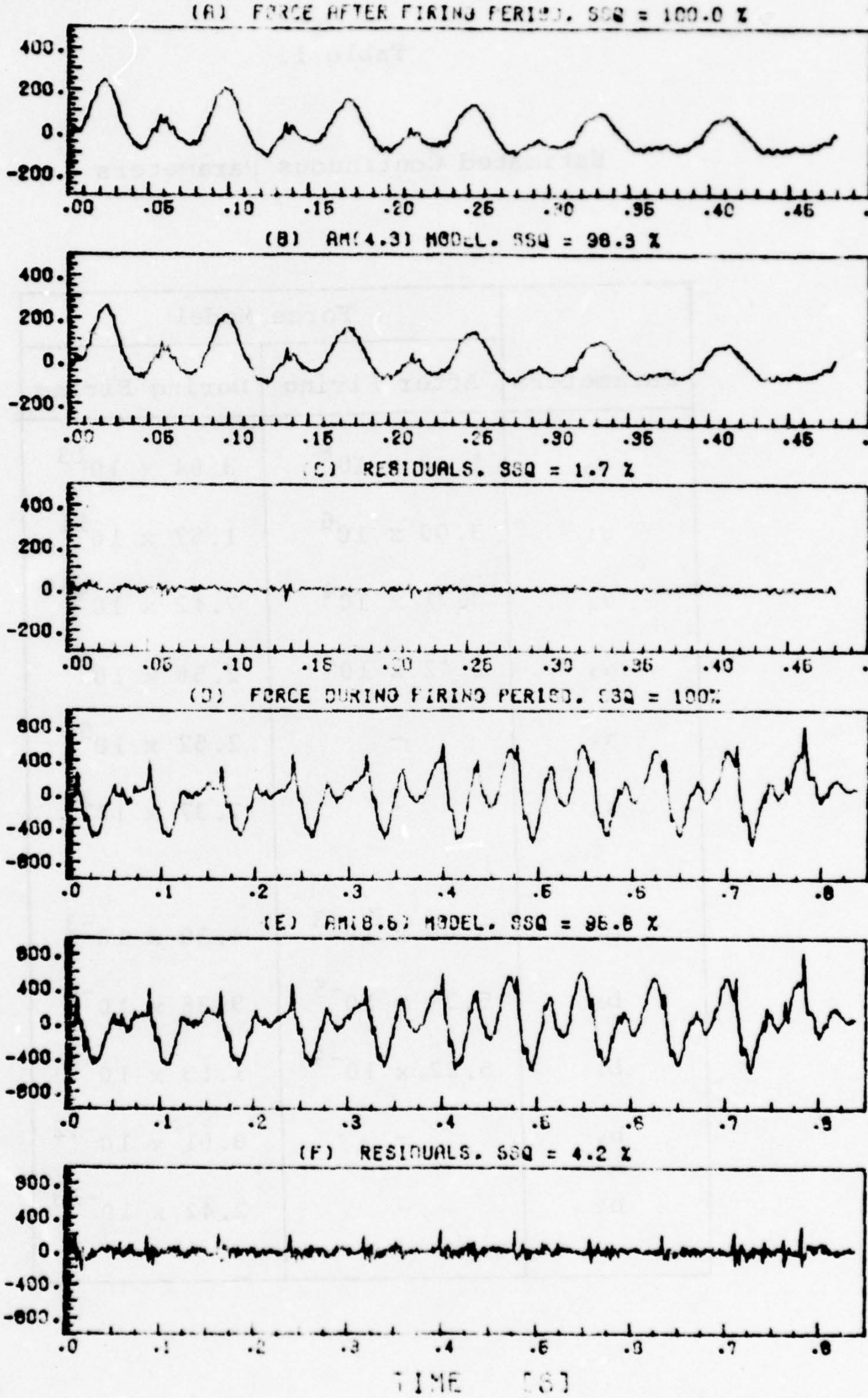

FIG. 5 FORCES DATA, MODEL AND RESIDUALS

Table 1.

Estimated Continuous Parameters

Parameters	Force Model	
	After Firing	During Firing
α_0	1.48×10^{8}	3.84×10^{13}
α_1	3.00×10^{6}	1.57×10^{11}
α_2	3.71×10^{4}	7.42×10^{9}
α_3	1.42×10^{2}	2.56×10^{7}
α_4	-	2.52×10^{5}
α_5	-	7.37×10^{2}
b_1	3.09×10^{-3}	4.30×10^{-3}
b_2	5.39×10^{-5}	9.38×10^{-6}
b_3	5.22×10^{-9}	1.13×10^{-8}
b_4	-	8.01×10^{-12}
b_5	-	2.42×10^{-13}

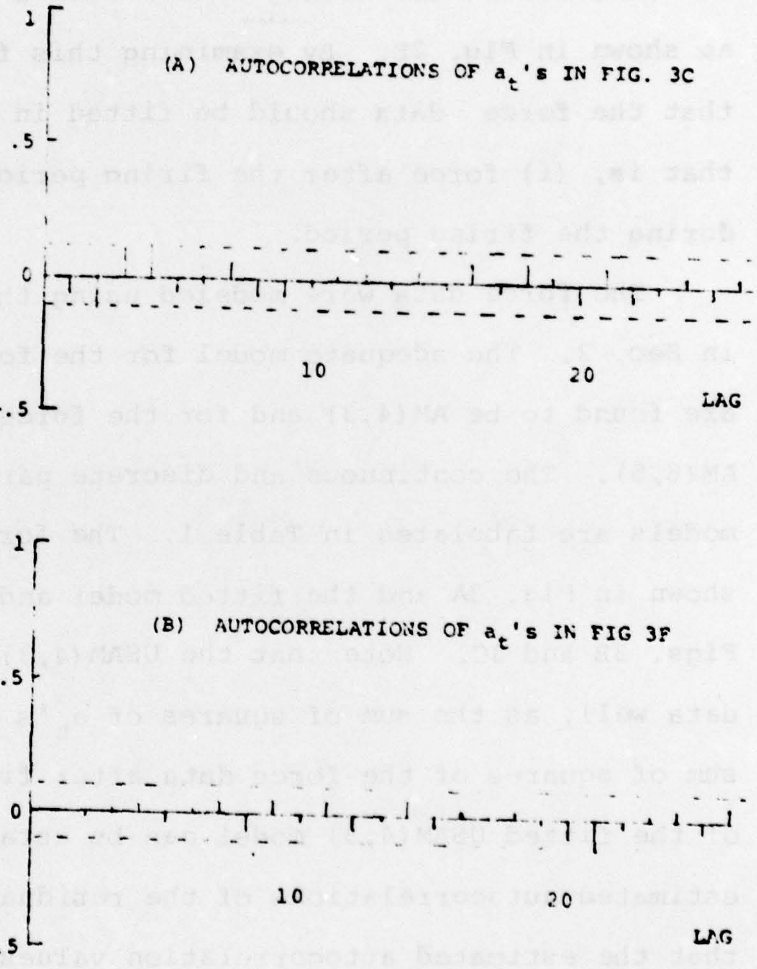

FIG. 4 SAMPLE AUTOCORRELATIONS OF a_t's

3. Modeling of Force Data

There are four sets of measured forces in a firing experiment. In order to reduce these force data, it is noted from Fig. 1 that the effective force is horizontal, hence the four forces are averaged to obtain a total force series as shown in Fig. 2E. By examining this figure, one can see that the force data should be fitted in two separate aspects; that is, (i) force after the firing period and (ii) force during the firing period.

The force data were modeled using the method summarized in Sec. 2. The adequate model for the force after firing are found to be AM(4,3) and for the force during firing are AM(6,5). The continuous and discrete parameters of the two models are tabulated in Table 1. The force after firing is shown in Fig. 3A and the fitted model and a_t's are shown in Figs. 3B and 3C. Note that the USAM(4,3) models fits the data well, as the sum of squares of a_t's is only 1.7% of the sum of squares of the force data after firing. The adequacy of the fitted USAM(4,3) model can be established from the estimated autocorrelations of the residuals. Figure 4A shows that the estimated autocorrelation values at various lags are below the 2σ limits (indicated by dotted lines). Therefore, the assumption that the a_t's are uncorrelated is not violated.

Figs. 3D-3F show the force during firing, the fitted USAM(6,5) model, and the a_t's. The adequacy of the fitted USAM(6,5) model is apparent from the estimated autocorrelations of a_t's shown in Fig. 4B. The sum of squares of a_t's being approximately 4.2% of the sum of squares of the data.

It should be noted that although the models fitted to the force data are continuous, their values can be estimated at discrete time instants only. For this reason, the models shown in Figs. 4b and 4e are referred to, respectively, as USAM(4,3) and USAM(6,5).

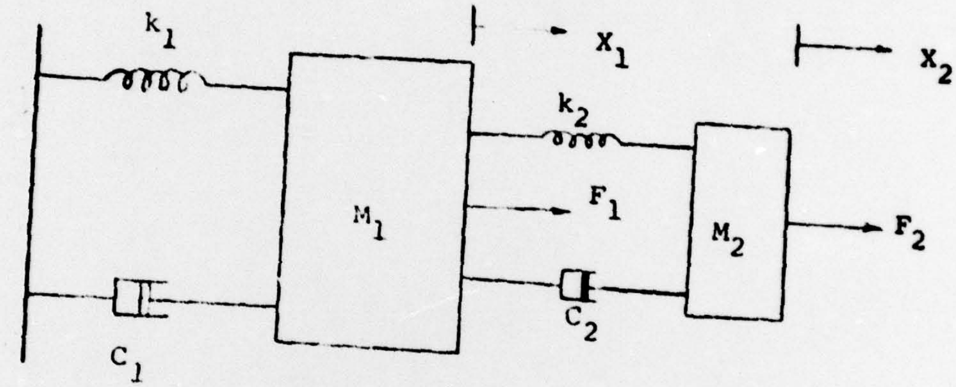

FIG. 5 TWO DEGREE-OF-FREEDOM MODEL

4. Decomposition of the Fitted AM Models

In order to associate the fitted AM(4,3) and AM(6,5) models with the firing experiment, they are decomposed into subsystems. Before the decomposition technique is introduced, it should be pointed out that when the gun is operating, the bolt assembly moves relative to the gun. The gun ceases to operate when the bolt assembly and the gun are locked together. The fitted models reveal this phenomenon, as the model fitted to the force after firing is of lower order than the model of the force during firing.

4.1 Decomposition of the Force Model After Firing

(a) Theoretic Deterministic Model

When the M-139 machine gun ceases to operate, the bolt assembly and the rest of the gun can be considered as one mass. The experiment then can be modeled as a two degree-of-freedom system as shown in Fig. 5. Here M_1, C_1 and k_1 represent the mass, damping coefficient and spring constant for the table, and M_2, C_2, and k_2 represent the recoil system. When this system is excited by forces F_1 and F_2, the system dynamic, designated by X_1 and X_2 can be written in terms of the Laplace transform variable as

$$\begin{bmatrix} [M_1 s^2+(C_1+C_2)s+k_1+k_2] & -(C_2 s+k_2) \\ -(C_2 s+k_2) & M_2 s^2+C_2 s+k_2 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix} \tag{9}$$

For convenience let

$$T_{11} = M_1 s^2 + (C_1 + C_2)s + k_1 + k_2 \tag{10}$$

$$T_{12} = C_2 s + k_2 \tag{11}$$

$$T_{22} = M_2 s^2 + C_2 s + k_2 \tag{12}$$

then

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} T_{22}F_1(s) + T_{12}F_2(s) \\ T_{12}F_1(s) + T_{11}F_2(s) \end{bmatrix} (T_{11}T_{22} - T_{12}^2)^{-1} \tag{13}$$

The measured force as a function of time t is

$$X(t) = C_2 \frac{d}{dt} [X_2(t) - X_1(t)] + k_2 [X_2(t) - X_1(t)] \tag{14}$$

The Laplace transform of X(t) is

$$X(s) = T_{12}[X_2(s) - X_1(s)] \tag{15}$$

therefore

$$(T_{11}T_{22} - T_{12}^2)\, X(s)$$
$$= T_{12}\, [(T_{12} - T_{22})F_1(s) + (T_{11} - T_{12})F_2(s)] \tag{16}$$

In time domain, Eq. (16) is of the form

$$(D^4 + \alpha_3 D^3 + \alpha_2 D^2 + \alpha_1 D + \alpha_0)\, X(t) =$$
$$\{ k_1 k_2 F_2(t) + D(k_1 C_2 + C_1 k_2)F_2(t) + D^2[(C_1 C_2 + M_1 k_2)F_2(t) - k_2 M_2 F_1(t)]$$
$$+ D^3[C_2(M_1 F_2(t) - M_2 F_1(t))]\} \,/\, M_1 M_2 \tag{17}$$

where

$$\alpha_0 = k_1k_2/M_1M_2 = \omega_1{}^2\omega_2{}^2 \tag{18}$$

$$\alpha_1 = k_1C_2/M_1M_2+k_2C_1/M_1M_2 = 2\zeta_1\omega_1\omega_2{}^2+2\zeta_2\omega_1{}^2\omega_2 \tag{19}$$

$$\alpha_2 = k_1/M_1+k_2/M_2+k_2/M_1+C_1C_2/M_1M_2 = 4\zeta_1\omega_1\zeta_2\omega_2 \tag{20}$$

$$+\omega_1{}^2+\omega_2{}^2(1+M_2/M_1)$$

$$\alpha_3 = C_1/M_1+C_2/M_2+C_2/M_1 = 2\zeta_1\omega_1+2\zeta_2\omega_2(1+M_2/M_1), \tag{21}$$

ζ_1, ω_1 and ζ_2, ω_2 are, respectively, the damping ratio and natural frequencies of the table and recoil system.

(b) Comparison Between the theoretic deterministic Model and the fitted AM(4,3) model.

The AM(4,3) model fitted to the force after firing is

$$(D^4 + \alpha_3D^3 + \alpha_2D^2 + \alpha_1D + \alpha_0)\ X(t)$$

$$= (1 + b_1D + b_2D^2 + b_3D^3)\ Z(t) \tag{22}$$

Note that the right hand side of Eq. (22) cannot be interpreted in terms of the parameters in the right hand side of Eq. (17) due to the unknown nature of $F_1(t)$ and $F_2(t)$. However, the left hand side of Eq. (17) and Eq. (22) are the same; therefore, Eqs. (18-21) also define α_0, α_1, α_2, and α_3 of Eq. (22) in terms of the M-139 firing experiment subsystem parameters.

Since the exact solutions for M_1, C_1, k_1, M_2, C_2, and k_2 using Eqs. (18-21) are not possible, an approximation method is developed. It can be seen from Eqs. (18-21) that if M_1 and M_2 are measured, then the values of ζ_1, ω_1, ζ_2 and ω_2 are identifiable. Since M_1 and M_2 are not known, the ratio M_2/M_1 is neglected using a priori knowledge that the mass of the gun is much smaller than the effective mass of the table. Equations (20) and (21) can now be written,

$$\alpha_2 = 4\zeta_1\omega_1\zeta_2\omega_2 + \omega_1^2 + \omega_2^2 \tag{23}$$

$$\alpha_3 = 2\zeta_1\omega_1 + 2\zeta_2\omega_2 \tag{24}$$

Now let's consider the autoregressive roots μ_1, μ_2, μ_3 and μ_4 such that

$$(D-\mu_1)(D-\mu_2)(D-\mu_3)(D-\mu_4) = (D^4+\alpha_3 D^3+\alpha_2 D^2+\alpha_1 D+\alpha_0) \tag{25}$$

It follows from Eq. (25) that

$$\begin{aligned}
\alpha_0 &= \mu_1\mu_2\mu_3\mu_4 \\
\alpha_1 &= -(\mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4) \\
\alpha_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 \\
\alpha_3 &= -(\mu_1 + \mu_2 + \mu_3 + \mu_4)
\end{aligned} \tag{26}$$

Table 2.

Estimates of Subsystems Parameters from Force Models.

Force Model	Supporting Table		Recoil		Bolt	
	ζ_1	ω_1 (Hz)	ζ_2	ω_2 (Hz)	ζ_3	ω_3 (Hz)
After Firing	.13	26.4	.69	11.7	---	---
During Firing	.004	25.9	.03	13.1	.79	73.8

Comparing Eqs. (18, 19, 23 and 24) with Eqs. (26), it can be seen that

$$\mu_1, \mu_2 = -\zeta_1\omega_1 \pm \omega_1 \ (\zeta_1^{\,2}-1)^{1/2}$$
$$\mu_3, \mu_4 = -\zeta_2\omega_2 \pm \omega_2 \ (\zeta_2^{\,2}-1)^{1/2} \qquad (27)$$

Therefore, the approximate values of the damping ratios and natural frequencies of the table and the recoil system can be found from the autoregressive roots of the fitted model. These values are given in Table 2.

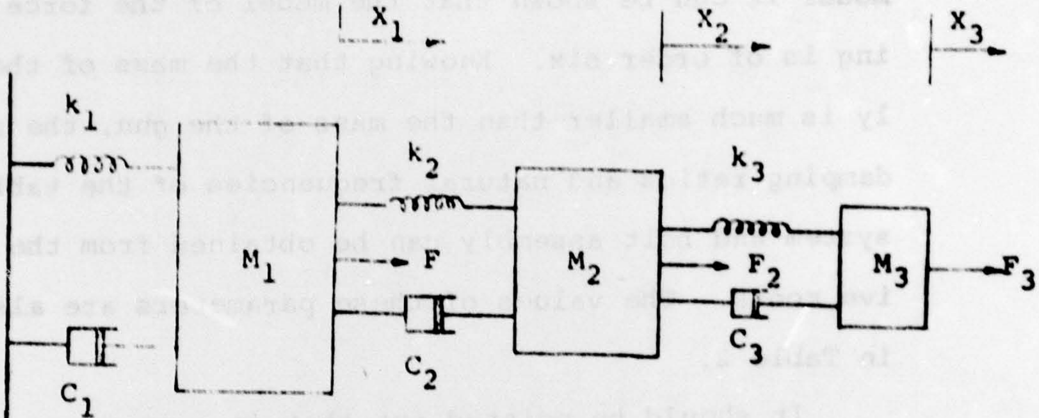

FIG. 6 THREE DEGREE-OF-FREEDOM MODEL

4.2 Decomposition of Force Model During Firing.

The force during firing is visualized to be from a three-degree of freedom spring, mass, and damper system as shown in Fig. 6. Here k_1, C_1 and M_1 represent the table, k_2, C_2 and M_2 represent the recoil system and k_3, C_3 and M_3 represent the bolt assembly. Using a similar approach to estimate the model it can be shown that the model of the force during firing is of order six. Knowing that the mass of the bolt assembly is much smaller than the mass of the gun, the approximate damping ratios and natural frequencies of the table, recoil system and bolt assembly can be obtained from the autoregressive roots. The values of these parameters are also given in Table 2.

It should be pointed out that in a previous paper a set of table displacement data from the same firing experiment has been analyzed. The estimates of the table's damping ratio and natural frequency were .176 and 25.7 Hz [4]. It can be seen from Table 2 that the estimates of the table's natural frequency, using models from force signals after and during firing are, 26.4 and 25.9 Hz, respectively. Although the force models are decomposed using an approximation method, the estimates of the table's natural frequency are well in agreement with the previous estimate of 25.7 Hz. Decompositions of force models after and during firing also yield estimates of the table's damping ratio. The values of these estimates, shown in Table 2, are .13 and .004 as compared with the previous estimate of .176.

5. Conclusions

(i) The force data were fitted in two parts: during and after the firing period. The adequate model for the force during firing is a continuous autoregressive moving average model of orders 6 & 5 [AM(6,5)], with the sum of squares of residuals approximately 4.2% of the total sum of squares. The model fitted to the force after firing is AM(4,3) with the residuals sum of squares estimated to be 1.7% of the total sum of squares.

(ii) The M-139 experiments are visualized to be composed of three subsystems. They are a table, a recoil system and a bolt assembly. Assuming that the table's mass is much larger than the gun's mass, and that the gun's mass is much larger than the mass of the bolt assembly, an analysis is made to decompose the force data models. As a result of this decomposition, the damping ratios and natural frequencies of the three subsystems were obtained.

(iii) The estimate of the table's natural frequency is 25.9 Hz when the model for the force during the firing period was used and 26.4 Hz when the model for the force after the firing period was used. These values are in agreement with the estimate of the table's natural frequency of 25.7 Hz using the model fitted to the table displacement data published earlier. Using the model for the force data during and after firing, the table's damping ratio is estimated as .004 and .13, respectively. The table's damping ratio obtained from the table displacement model was .176.

6. Acknowledgement

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